ON A LIOUVILLE TRANSFORMATION FOR
$$u_{xx} + u_{yy} \pm a^2(x,y)u = 0$$

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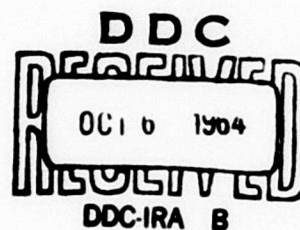
Summary

It is shown that the equation $u_{xx} + u_{yy} \pm a^2(x,y)u = 0$ may be reduced to the form $u_{ss} + u_{tt} \pm u = 0$ by a change of variable of the form $s = s(x,y)$, $t = t(x,y)$, provided that $\log a(x,y)$ is a harmonic function.

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1. Introduction

In the study of the boundedness, stability and asymptotic behavior of the solutions of the second-order linear differential equation

(1) $$u'' \pm a^2(t)u = 0,$$

an essential tool is the Liouville transformation

(2) $$s = \int_0^t a(t_1)dt_1.$$

This is a $1-1$ transformation for large t if $a(t) > 0$ for all $t \geq t_0$. It transforms (1) into the equation

(3) $$\frac{d^2u}{ds^2} + \frac{a'(t)}{a^2(t)} \frac{du}{ds} \pm u = 0.$$

If $a'(t)/a^2(t)$ is small in some sense, either as $t \longrightarrow \infty$ because of the rate of increase of $a(t)$, or because $a(t)$ is slowly varying, we have an equation with almost-constant coefficients. A further change of variable

(4) $$u = v/a(t)^{1/2}$$

reduces (3) to an equation of the form

(5) $$\frac{d^2v}{ds^2} + (\pm 1 + b(s))v = 0.$$

From this equation, the WKB approximation follows immediately. For tne details of these transformations and many further results, see Chapter 6 of our book, [1].

In studying the asymptotic behavior of the solutions of partial differential equations of the form

$$(6) \qquad u_{xx} + u_{yy} \pm a^2(x,y)u = 0$$

as $x,y \longrightarrow \infty$, it is tempting to search for a transformation similar to that given in (2). In this paper, we will show that the desired transformation exists, and indeed does more than what might be expected, under certain favorable circumstances.

2. Preliminary Calculations

Replacing x and y by two as yet unspecified independent variables s and t, we obtain the relations

$$(1) \qquad u_{xx} = u_{ss}s_x^2 + 2u_{st}s_x t_x + u_{tt}t_x^2 + u_s s_{xx} + u_t t_{xx}$$

$$u_{yy} = u_{ss}s_y^2 + 2u_{st}s_y t_y + u_{tt}t_y^2 + u_s s_{yy} + u_t t_{yy}.$$

The equation of (1.6) in the new variables has the form

$$(2) \qquad u_{ss}(s_x^2 + s_y^2) + u_{tt}(t_x^2 + t_y^2) + 2u_{st}(s_x t_x + s_y t_y)$$

$$+ u_s(s_{xx} + s_{yy}) + u_t(t_{xx} + t_{yy})$$

$$\pm a^2(x,y)u = 0.$$

We wish to determine two functions of x and y, $s(x,y)$ and $t(x,y)$, such that the following relations hold:

(3)
$$s_x^2 + s_y^2 = a^2(x,y)$$

$$t_x^2 + t_y^2 = a^2(x,y)$$

$$s_x t_x + s_y t_y = 0.$$

From the first two of these relations, we see that

(4)
$$s_x = a(x,y) \cos \phi, \quad t_x = a(x,y) \cos \psi,$$

$$s_y = a(x,y) \sin \phi, \quad t_y = a(x,y) \sin \psi,$$

for two functions $\phi(x,y)$ and $\psi(x,y)$.

The third relation, the orthogonality relation, requires that

(5)
$$\phi - \psi = \pm \pi/2.$$

Choosing $\psi = \phi + \pi/2$, we have

(6)
$$s_x = a(x,y) \cos \phi, \quad t_x = a(x,y) \sin \phi,$$

$$s_y = a(x,y) \sin \phi, \quad t_y = - a(x,y) \cos \phi.$$

The Jacobian of the transformation

(7)
$$s = s(x,y)$$

$$t = t(x,y)$$

is thus $a^2(x,y)$. It follows that we wish to assume that $a^2(x,y) > 0$ for $x \geq x_0$, $y \geq y_0$.

It remains to determine under what conditions upon $a(x,y)$ there exists a function $\phi(x,y)$ satisfying the desired relations.

3. Condition Upon $a(x,y)$

In order for (2.6) to hold, we must have

(1) $\qquad (s_x)_y = (s_y)_x, \quad (t_x)_y = (t_y)_x,$

or

(2) $\qquad (a \cos \phi)_y = (a \sin \phi)_x, \quad (a \sin \phi)_y = (-a \cos \phi)_x.$

A simple calculation shows that (2) is equivalent to the relations

(3) $\qquad \phi_x = a_y/a = \frac{\partial}{\partial y} (\log a),$

$\qquad \phi_y = -a_x/a = -\frac{\partial}{\partial x} (\log a).$

It follows that ϕ exists if, and only if,

(4) $\qquad \frac{\partial^2}{\partial x^2} (\log a) + \frac{\partial^2}{\partial y^2} (\log a) = 0.$

In other words, we must suppose that $\log a(x,y)$ is a harmonic function.

4. The Transformed Equation

Having made this assumption, we reap the bonus that

$$(1) \qquad s_{xx} = a_x \cos \phi - a\phi_x \sin \phi = a_x \cos \phi - a_y \sin \phi,$$

$$s_{yy} = a_y \sin \phi + a\phi_y \cos \phi = a_y \sin \phi - a_x \cos \phi,$$

whence

$$(2) \qquad s_{xx} + s_{yy} = 0,$$

and similarly

$$(3) \qquad t_{xx} + t_{yy} = 0.$$

Thus the equation in terms of the new variables is

$$(4) \qquad u_{ss} + u_{tt} \pm u = 0.$$

5. Discussion

In connection with the study of the asymptotic behavior of the solutions of (1.6), we need only demand that the principal term of $\log a(x,y)$ be harmonic. Using this principal term and carrying through the foregoing transformation, we will obtain an equation of the form

$$(1) \qquad u_{ss} + u_{tt} + (\pm 1 + h(s,t))u = 0.$$

We shall discuss these matters elsewhere.

Observe that for $a(x,y) = e^{kxy}$, the foregoing trans-

formation yields various reasonably explicit solutions of the equation

(2) $\qquad u_{xx} + u_{yy} \pm e^{2kxy} u = 0.$

Reference

1. Bellman, R., Stability Theory of Differential Equations,
    McGraw-Hill Book Company, Inc., New York, 1954.

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